Fast Positive-Real Balanced Truncation of Symmetric Systems Using Cross Riccati Equations

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Abstract

We present a computationally efficient implementation of positive-real balanced truncation (PRBT) for symmetric multiple-input multiple-output (MIMO) systems. The solution of a pair of algebraic Riccati equations (AREs) in conventional PRBT, whose complexity limits practical large-scale realization, is replaced with the solution of one cross Riccati equation (XRE). The cross-Riccatian solution then permits simple construction of projection matrices without actually balancing the system. The method encompasses passive linear networks, as commonly used in interconnect and package modelings, due to their inherent reciprocity and therefore symmetric transfer functions. Effectiveness of the proposed approach is verified by numerical examples.

1. Introduction

Parasitic extraction of packages and deep-submicron interconnects produces massively coupled RLC elements that prohibit direct computer simulation. Model order reduction (MOR) has become a standard routine whereby the initial model is approximated by a reduced-order model with little loss in time/frequency-domain accuracy. Moreover, the reduced-order model must preserve stability and passivity of the original model to ensure valid global simulation [3, 8, 9]. Specifically, a passive system is one that does not generate energy internally. A strictly passive system is dissipative and is automatically stable. In linear systems, passivity is equivalent to positive realness [3].

Among numerous MOR approaches, the control-theoretic balanced truncation (BT) schemes are known to have superior accuracy and closed-form error bounds [1–3, 9, 13–15]. The key idea of BT is to align and sort the internal states of the original model based on their participation in input-output state/energy transfer. The least important states are then truncated with little impact on the system responses. In standard BT, the bottleneck is the solution of two linear matrix equations, called Lyapunov equations, for obtaining the controllability and observability Gramians. The cross product of these Gramians is then used to obtain low-order projection matrices to reduce (truncate) the original system. However, standard BT does not necessarily preserve passivity. Positive-real balanced truncation (PRBT), also known as positive-real truncated balanced realization (PR-TBR) [9], is another BT approach that preserves both passivity and stability, and has no special structural restriction on the internal state space. However, it involves the solution of a pair of quadratic matrix equations, called algebraic Riccati equations (AREs), whose complexity is even higher than that of Lyapunov equations [13–15].

In the standard BT of symmetric multiple-input multiple-output (MIMO) systems\(^1\), information in the cross product of controllability and observability Gramians can be directly extracted from one Sylvester equation which is a linear matrix equation [1, 2, 5]. As only one matrix equation is solved, computation is effectively halved, with further advantages like better consistency and numerical robustness [1, 2]. A quadratic counterpart of the Sylvester equation, called the cross Riccati equation (XRE), appeared in [6,10] on control topics like feedback control and discrete stochastic processes. Its integration with PRBT, however, has not been elaborated nor fully appreciated by the EDA community.

This paper generalizes the cross-Gramian framework in standard BT [1] to the cross-Riccatian framework for PRBT of symmetric MIMO systems. An invariant subspace method for solving the XRE is described. A Schur decomposition procedure, borrowed from the standard BT scenario [1], then allows simple construction of projection matrices to obtain equivalent PRBT-reduced models with-

\(^1\)Hence all single-input single-output (SISO) systems are automatically included.
out actually balancing the system. The scope of the algorithm encompasses passive linear circuits such as RLC networks, commonly used in interconnect and package modelings, which exhibit reciprocity and therefore have symmetric transfer functions [7, 11]. (A system is reciprocal if its admittance or impedance matrix is symmetric, i.e., the voltage and current at any two points in the network can be interchanged.) Numerical examples then confirm the remarkable efficiency of the proposed method over conventional PRBT realizations.

2. Background and Preliminaries

2.1. Positive-Real Balanced Truncation

Starting with the state space of a minimal (but not necessarily symmetric) MIMO square system

\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad (1a) \\
y &= Cx + Du, \quad (1b)
\end{align*}
\]

where \(A \in \mathbb{R}^{n \times n}, B, C^T \in \mathbb{R}^{n \times m}, D \in \mathbb{R}^{m \times m}\), and \(u, y\) are power-conjugate: for every entry of \(u\) that is a node voltage (branch current), the corresponding entry of \(y\) is a branch current (node voltage) such that \(u^T y\) represents a power metric. \(A\) is stable or its spectrum is in the left open half plane, denoted by \(\text{spec}(A) \subseteq \mathbb{C}_-\). Let \(M > 0\) denote a positive definite (positive semidefinite) matrix \(M\), we assume without loss of generality that \(D + D^T > 0\), otherwise the reduction technique in [12] is used to achieve this. Also, an impulse-free system in the descriptor representation [3] with a singular \(E\) before \(\dot{x}\) can be put into the regular form in (1) [9]. The positive-real lemma [3] states that the linear system (1) is passive if and only if there exists an \(X \in \mathbb{R}^{n \times n}\) satisfying the linear matrix inequality

\[
\begin{bmatrix}
A^T X + X A & X B - C^T \\
B^T X - C & -(D + D^T)
\end{bmatrix} \leq 0. \quad (2)
\]

Applying Schur complement on (2), PRBT formulates and solves for the unique stabilizing solutions, \(X_c(\geq 0)\) and \(X_o(\geq 0)\), to the dual AREs

\[
\begin{align*}
AX_c + X_c A^T + (X_c C^T - B)(D + D^T)^{-1}(C X_c - B^T) &= 0, \quad (3a) \\
A^T X_o + X_o A + (X_o B - C^T)(D + D^T)^{-1}(B^T X_o - C) &= 0. \quad (3b)
\end{align*}
\]

Existence of solutions is guaranteed by the passivity assumption. We call \(X_c\) the controllability Riccatian and \(X_o\) the observability Riccatian. Factoring out \(X_c\) and \(X_o\) in (3) for their coefficient matrices, we define

\[
\begin{align*}
A_c &= A - (B - X_c C^T)(D + D^T)^{-1}C, \quad (4a) \\
A_o &= A - B(D + D^T)^{-1}(C - B^T X_o). \quad (4b)
\end{align*}
\]

Stabilizability of \(X_c\) and \(X_o\) implies \(\text{spec}(A_c) \subset \mathbb{C}_-\) [15]. Let \(X_c = L_c L_c^T\) and \(X_o = L_o L_o^T\), where \(L_c, L_o \in \mathbb{R}^{n \times n}\), be any matrix square-root decompositions. Compute the singular value decomposition (SVD)

\[
L_c^T L_o = U \Sigma V^T, \quad \Sigma = \text{diag}(\sigma_1, \cdots, \sigma_n). \quad (5)
\]

Here \(\Sigma\) is a diagonal matrix and

\[
\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \gg \sigma_{r+1} \geq \cdots \geq \sigma_n. \quad (6)
\]

Define the matrices

\[
T_R = L_c U \Sigma^{-\frac{1}{2}} \quad \text{and} \quad T_L = T_R^{-1} = \Sigma^{-\frac{1}{2}} V^T L_o^T, \quad (7)
\]

and using ‘\(\rightarrow\)’ to denote the corresponding similarity transform, we get the positive-real-balanced model

\[
(A, B, C, D) \rightarrow (\hat{A}, \hat{B}, \hat{C}, D) := (T_L A T_R, T_L B, C T_R, D). \quad (8)
\]

Such transform also results in simultaneously diagonalized Riccatians in the new state space, namely

\[
X_c \rightarrow T_L X_c T_L^T = \Sigma \quad \text{and} \quad X_o \rightarrow T_R^T X_o T_R = \Sigma, \quad (9)
\]

satisfying

\[
\hat{A} \Sigma + \Sigma \hat{A}^T + (\Sigma \hat{C}^T - \hat{B})(D + D^T)^{-1}(\hat{C} \Sigma - \hat{B}^T) = 0, \quad (10a) \\
\hat{A}^T \Sigma + \Sigma \hat{A} + (\Sigma \hat{B} - \hat{C}^T)(D + D^T)^{-1}(\hat{B}^T \Sigma - \hat{C}) = 0. \quad (10b)
\]

The states in \((\hat{A}, \hat{B}, \hat{C}, \hat{D})\) are aligned in descending importance in energy transfer [9]. Express \(\Sigma = \text{diag} (\Sigma_b, \Sigma_s)\) where \(\Sigma_b \in \mathbb{R}^{r \times r}\) holds the “bigger” singular values and \(\Sigma_s \in \mathbb{R}^{(n-r) \times (n-r)}\) holds the “smaller” ones. Partition columns of \(T_R\) and rows of \(T_L\) accordingly so that with respect to (9),

\[
\begin{align*}
X_c &= T_R \Sigma T_R^T = \begin{bmatrix} T_{Rb} & T_{Rs} \end{bmatrix} \begin{bmatrix} \Sigma_b \\ \Sigma_s \end{bmatrix} \begin{bmatrix} T_{Rb}^T \\ T_{Rs}^T \end{bmatrix}, \quad (11a) \\
X_o &= T_L^T \Sigma T_L = \begin{bmatrix} T_{Lb}^T \\ T_{Ls}^T \end{bmatrix} \begin{bmatrix} \Sigma_b \\ \Sigma_s \end{bmatrix} \begin{bmatrix} T_{Lb} \\ T_{Ls} \end{bmatrix}. \quad (11b)
\end{align*}
\]

The PRBT-reduced model is obtained from the rank-\(r\) subspace projection

\[
(\hat{A}_r, \hat{B}_r, \hat{C}_r, \hat{D}) := (T_{Lb} A T_{Rb}, T_{Lb} B, C_T R, D), \quad (12)
\]

where \(\hat{A}_r \in \mathbb{R}^{r \times r}, \hat{B}_r, \hat{C}_r \in \mathbb{R}^{r \times m}\). The system in (12) is passive and stable and the transfer matrix \(G_r(s) = D + \hat{C}_r(s I - \hat{A}_r)^{-1} \hat{B}_r\) has an \(H_\infty\)-norm error bound [9] with respect to \(G(s) = D + C(s I - A)^{-1} B\).
2.2. Sylvester Equation in the Standard BT of Symmetric Systems

Standard BT requires solving a pair of dual Lyapunov equations
\[
AW_c + W_c A^T + BB^T = 0, \quad (13a)
\]
\[
A^T W_o + W_o A + C^T C = 0, \quad (13b)
\]
whose solutions, \( W_c \) and \( W_o \), are the controllability and observability Gramians, respectively. The rest of the standard BT procedure is the same as in PRBT except that \( X_c \) and \( X_o \) are replaced with \( W_c \) and \( W_o \), respectively. Previous work has studied the use of the cross Gramian in standard BT of symmetric MIMO (thus also SISO) systems [2]. Specifically, for a symmetric system, the cross Gramian, \( W_{co} \), is solved from the Sylvester equation
\[
AW_{co} + W_{co} A + BC = 0, \quad (14)
\]
where \( W_{co} \) satisfies the important property \( W_{co}^2 = W_c W_o \). Since \( W_{co} \) contains both the controllability and observability information, standard BT can directly make use of the eigenvector bases of \( W_{co} \) without having to solve two Lyapunov equations [1].

3. PRBT of Symmetric Systems

We first define system symmetry. The MIMO system \((A, B, C, D)\) in (1) is symmetric if \( G(s) = G(s)^T \) or
\[
D + C(sI - A)^{-1} B = D^T + B^T (sI - A)^{-1} C^T, \quad (15)
\]
for all \( s \in \mathbb{C} \setminus \text{spec}(A) \) where ‘\(-\)’ denotes set subtraction. This necessitates \( D = D^T \). Symmetry and minimality of \((A, B, C)\) means that it is similar to \((A^T, C^T, B^T)\) through a unique nonsingular \( T \in \mathbb{R}^{n \times n} \) (e.g., [6]) such that
\[
A^T = T^{-1} A T, \quad C^T = T^{-1} B, \quad B^T = C T. \quad (16)
\]
Analogous to (13) and (14), we formulate, with respect to (3), the cross Riccati equation (XRE)
\[
AX_{co} + X_{co} A + (X_{co} B - B) (D + D^T)^{-1} (C X_{co} - C) = 0, \quad (17)
\]
Here \( X_{co} \) is called the cross Riccatian as in [10]. The following describes a two-step cross-Riccatian PRBT flow for symmetric systems.

3.1. Solving the Cross Riccati Equation

Based on (16) and the results in [6], we present a possible way of solving (17) associated with a symmetric system.

First, define the two Hamiltonian matrices, \( H_c \) and \( H_o \), corresponding to (3a) and (3b), respectively,
\[
H_c = \begin{bmatrix} A^T & 0 \\ 0 & -A \end{bmatrix} - \begin{bmatrix} C^T \\ B \end{bmatrix} (D + D^T)^{-1} \begin{bmatrix} B^T & -C \end{bmatrix}, \quad (18a)
\]
\[
H_o = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} - \begin{bmatrix} B \\ C^T \end{bmatrix} (D + D^T)^{-1} \begin{bmatrix} C & -B^T \end{bmatrix}. \quad (18b)
\]
Spectral structure of a Hamiltonian matrix and solutions of (3a) and (3b) by identifying the stable invariant subspaces of (18a) and (18b), respectively, are well studied, e.g., [15]. We note that \( H_c \) and \( H_o \) share the same spectrum since
\[
H_c = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} (-H_o) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}. \quad (19)
\]
In [15] it has been shown that when the stable and unstable subspaces of \( H_o \) are separated (passivity implies no purely imaginary eigenvalues), i.e.,
\[
H_o \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \begin{bmatrix} \Phi_- & 0 \\ 0 & \Phi_+ \end{bmatrix}, \quad (20)
\]
with \( \Phi_-, \Phi_+ \in \mathbb{R}^{n \times n} \) corresponding to the stable and unstable parts, respectively, of \( \text{spec}(H_o) \), then the stabilizing solution to (3b) is \( X_o = X_{21} X_{11}^{-1} \) and that to (3a) is \( X_c = X_{12} X_{22}^{-1} \). Now we show that this invariant subspace approach is also applicable to (17). First, define
\[
H_{co} = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} - \begin{bmatrix} B \\ B \end{bmatrix} (D + D^T)^{-1} \begin{bmatrix} C & -C \end{bmatrix}, \quad (21)
\]
it can be seen that \( \text{spec}(H_{co}) = \text{spec}(H_o) = \text{spec}(H_c) \) since
\[
H_{co} = \begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} H_o \begin{bmatrix} I & 0 \\ 0 & T^{-1} \end{bmatrix}. \quad (22)
\]
Applying (21) to (19) we get
\[
H_{co} \begin{bmatrix} X_{11} & X_{12} \\ TX_{21} & TX_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ TX_{21} & TX_{22} \end{bmatrix} \begin{bmatrix} \Phi_- & 0 \\ 0 & \Phi_+ \end{bmatrix}. \quad (23)
\]
Then, noting
\[
\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} H_{co} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = -H_{co}, \quad (22)
\]
(22) can be broken into two equations
\[
H_{co} \begin{bmatrix} X_{11} \\ TX_{21} \end{bmatrix} = \begin{bmatrix} X_{11} \\ TX_{21} \end{bmatrix} \Phi_-, \quad (23a)
\]
\[
H_{co} \begin{bmatrix} TX_{22} \\ X_{12} \end{bmatrix} = \begin{bmatrix} TX_{22} \\ X_{12} \end{bmatrix} (-\Phi_+). \quad (23b)
\]
It is easily shown that both \( TX_{21} X_{11}^{-1} = TX_o \) and \( X_{12} X_{22}^{-1} T^{-1} = X_c T^{-1} \) solve (17). From the Hamiltonian structure, we further have \( \text{spec}(\Phi_-) = \text{spec}(-\Phi_+) \). Using \( \text{span}(\cdot) \) to denote the span (image) of a matrix, we have
\[
\text{span} \left( \begin{bmatrix} X_{11} \\ TX_{21} \end{bmatrix} \right) = \text{span} \left( \begin{bmatrix} TX_{22} \\ X_{12} \end{bmatrix} \right), \quad (24)
\]
which implies $TX_o = X_cT^{-1} =: X_{co}$ is a solution to (17) found by identifying the stable invariant subspace of $H_{co}$. The choice of the actual basis is immaterial as long as the spectrum restricted to it is the same, and from this the uniqueness of $X_{co}$ follows. Recalling (3) and (4), we generalize “stabilizability” in this symmetric-system cross-Riccatian sense: factoring out $X_{co}$ from both sides in (17) for the associated matrices, and noting $X_{co} = X_cT^{-1} = TX_o$ and (16), we get

\[
A - (B - X_{co}B)(D + D^T)^{-1}C = A_c,
\]

\[
A - B(D + D^T)^{-1}(C - CX_{co}) = A_o.
\]

(25a)

(25b)

It is readily seen that $\text{spec}(A_c) = \text{spec}(A_o) = \text{spec}(\Phi_-) \subset \mathbb{C}_-$. Also, similar to results in Section 2.2, we have

\[
X_{co}^2 = X_cX_{co},
\]

(26)

and the actual $T$ in (16) is immaterial. With the same notion as in Section 2.1, the effect of similarity transform on $X_{co}$ (in fact for arbitrary $T_R$ and $T_L = T_R^{-1}$) is

\[
X_{co} \rightarrow T_LT_{co}T_R,
\]

(27)

so $\text{spec}(X_{co})$ is invariant. Since there exists a similar system in which $X_c$ and $X_o$ are both diagonal [c.f. (9)], we have

\[
|\lambda_i(X_{co})| = \sigma_i, \quad i = 1, 2, \ldots, n,
\]

(28)

where $\sigma_i$s are the singular values in (6) and $\lambda_i($ denotes the $i$th eigenvalue in descending magnitude without loss of generality.

### 3.2. Constructing the Projection Matrices

The system balancing in PRBT, corresponding to (5)-(8), may sometimes be numerically inefficient and ill-conditioned. This is especially so for large-scale systems with nearly singular $X_c$ and/or $X_o$, i.e., some states are nearly uncontrollable and/or unobservable [1,2]. To avoid ill-conditioned arithmetic, we borrow results from [1] on standard BT and present an alternative way for obtaining a reduced-order model with the same transfer function as the PRBT-reduced model. From (26),

\[
\text{span}(X_{co}) = \text{span}(X_c),
\]

(29a)

\[
\text{span}(X_{co}^T) = \text{span}(X_o).
\]

(29b)

Therefore, $X_{co}$ contains both the controllable and observable subspaces in the cross-Riccatian sense. By (11) and (26),

\[
X_{co}^2 = \begin{bmatrix} T_{Rb} & T_{Rs} \end{bmatrix} \begin{bmatrix} \Sigma_b^2 & 0 \\ 0 & \Sigma_s^2 \end{bmatrix} \begin{bmatrix} T_{Lb} \\ T_{Ls} \end{bmatrix},
\]

(30)

However, solution of $X_{co}$ in (17) is generally not block-diagonal. To block-diagonalize $X_{co}$, we first compute an intermediate ordered real Schur form of it, namely,

\[
X_{co} = \begin{bmatrix} Q_b & Q_s \\ Q_s^T & \Omega \end{bmatrix} \begin{bmatrix} X_{co}^b & 0 \\ 0 & X_{co}^s \end{bmatrix} \begin{bmatrix} Q_s^T \\ Q_s \end{bmatrix},
\]

(31)

where $X_{co}^b \in \mathbb{R}^{r \times r}$ and $X_{co}^s \in \mathbb{R}^{(n-r) \times (n-r)}$ hold, in terms of magnitude, the bigger and smaller eigenvalues, respectively [c.f. (28)]. And $X_{co}^b$ and $X_{co}^s$ are block upper-triangular. Next, solve for $\Gamma \in \mathbb{R}^{r \times (n-r)}$ in the Sylvester equation

\[
X_{co}^b \Gamma - \Gamma X_{co}^s + \Omega = 0.
\]

(32)

It can be easily verified that

\[
X_{co} = \begin{bmatrix} Q_b & Q_s \\ Q_b \Gamma + Q_s \end{bmatrix} \begin{bmatrix} X_{co}^b & 0 \\ 0 & X_{co}^s \end{bmatrix} \begin{bmatrix} Q_s^T - \Gamma Q_s^T \\ Q_s \end{bmatrix} =: \begin{bmatrix} V_b & V_s \end{bmatrix} \begin{bmatrix} X_{co}^b & 0 \\ 0 & X_{co}^s \end{bmatrix} \begin{bmatrix} W_b \\ W_s \end{bmatrix},
\]

(33)

where $\begin{bmatrix} V_b & V_s \end{bmatrix}^{-1} = \begin{bmatrix} W_b \\ W_s \end{bmatrix}$. Comparing (33) and (30), we have that $T_{Rb}$ and $V_b$ span the same (right) eigenspace corresponding to $X_{co}^b$, while $T_{Rs}$ and $V_s$ span the same (right) eigenspace corresponding to $X_{co}^s$. Therefore, there exist nonsingular $M_1 \in \mathbb{R}^{r \times r}$ and $M_2 \in \mathbb{R}^{(n-r) \times (n-r)}$ such that

\[
\begin{bmatrix} T_{Rb} & T_{Rs} \end{bmatrix} = \begin{bmatrix} V_b & V_s \end{bmatrix} \begin{bmatrix} M_1 \\ 0 \\ 0 \\ M_2 \end{bmatrix}.
\]

(34)

Taking inverse on both sides, we also get

\[
\begin{bmatrix} T_{Lb} \\ T_{Ls} \end{bmatrix} = \begin{bmatrix} M_1^{-1} \\ 0 \\ 0 \\ M_2^{-1} \end{bmatrix} \begin{bmatrix} W_b \\ W_s \end{bmatrix}.
\]

(35)

Using (34) and (35), the reduced-order model by projection onto the “more significant” subspaces of (33),

\[
(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, D) = (W_bAV_b, W_bB, CV_b, D),
\]

(36)

is easily shown to be similar to (12), thereby producing the same transfer function as the PRBT-reduced model (so the $H_\infty$-norm error bound [9] accompanying PRBT still applies). In other words, an equivalent PRBT-reduced model can be obtained from the block diagonalization of $X_{co}$ without actually balancing the system.

### 4. Numerical Examples

We compare the proposed XRE-based PRBT method with conventional realizations in which two AREs are solved. All experiments are done in Matlab 7.0.4 on a 3GHz
The brackets are time breakdowns. From top: time for solving each ARE (only one in the last column), the last bracket is the time for matrix factorizations.

Table 1. CPU times (sec) for different PRBT implementations.

<table>
<thead>
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<th>slcares</th>
<th>aresolv (eigen)</th>
<th>XRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spiral inductor</td>
<td>(4.33)†</td>
<td>(3.83)</td>
<td>(4.00)</td>
</tr>
<tr>
<td>order=500</td>
<td>(3.89)</td>
<td>(3.98)</td>
<td>(0.52)</td>
</tr>
<tr>
<td></td>
<td>(0.52)</td>
<td>(5.03)</td>
<td>(0.56)</td>
</tr>
<tr>
<td>RLC ladder</td>
<td>(182.61)</td>
<td>(193.58)</td>
<td>(194.11)</td>
</tr>
<tr>
<td>order=800</td>
<td>(195.72)</td>
<td>(204.20)</td>
<td>(27.38)</td>
</tr>
<tr>
<td></td>
<td>(17.02)</td>
<td>(124.38)</td>
<td>(221.48)</td>
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<td>(17.02)</td>
<td>(124.38)</td>
<td>(221.48)</td>
</tr>
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</table>

We try out two real-life symmetric-system benchmarks available from the literature [15]: the first one is a spiral inductor of order 500; the second one is an RLC ladder of order 800. The CPU times for PRBT are tabulated in Table 1. As expected, the cross-Riccatian approach easily outperforms the conventional ones. This is because the competing ARE solvers are also based on stable invariant subspace identification, and that instead of two AREs, only one XRE needs to be solved in the proposed method. Figs. 1(a) and 2(a) show the frequency responses of the original systems and the reduced-order models, while Figs. 1(b) and 2(b) show the relative errors. The PRIMA [8] curves are also shown to highlight the superior accuracy of PRBT. We note in passing that PRIMA constitutes a passivity-preserving projection-type MOR scheme and poses certain structural constraints on the internal state space for which PRBT does not [9]. It is seen that curves from the cross-Riccatian approach apparently overlap with those from conventional PRBT. This is expected because the system obtained in (36) is similar to that in (12), but without the need of the balancing operation.

Some additional remarks are in order:
1. Referring to Sections 3.1 and 3.2, the invariant subspace approach for solving the XRE, without exploiting any matrix structure like sparsity or bands, has an \( O(n^3) \) complexity and an \( O(n^2) \) memory requirement. The workload arises mainly from the (ordered) Schur decompositions required in (23a) or (23b)] and (31), and the solution of the Sylvester equation in (32).

2. With reference to Table 1, a recent algorithm in [15] can almost halve the time in solving two AREs through completely separating the stable and unstable invariant subspaces of a Hamiltonian matrix. However, such approach still requires the time-consuming full matrix factorizations afterward. The XRE method, besides practically halving the time in ARE solution, further saves time by replacing several large-size matrix factorizations with only one matrix block diagonalization. In fact, from other experiments not reported here, the latter PRBT approach is consistently about 40% faster than the former (with both algorithms coded and run in Matlab). Another merit gained from the XRE approach is the better numerical consistency and accuracy as noted in [1, 2].

3. The cross-Riccatian framework is generic and any fast solver, not necessarily using the invariant subspace approach, may be adopted for computing \( X_{oo} \) in (17). For example, a recent ARE solver algorithm known as the quadratic alternating direction implicit (QADI) iteration [13, 14] may be modified to solve (17). A low-rank Cholesky-factor variant of QADI (called CFQADI) for XREs, if exists, can accelerate PRBT to a speed comparable to that of PRIMA [13, 14]. Research is being done along this direction and findings will be reported elsewhere.

5. Conclusion

This paper has presented an efficient PRBT implementation for symmetric MIMO systems. Instead of two AREs as in conventional PRBT, only one cross Riccati equation (XRE) needs to be solved. An invariant subspace method has been described for solving the XRE, followed by simple construction of projection matrices through block diagonalization. No positive-real balancing is required, thus avoiding the possible numerical ill-conditions in nearly uncontrollable/unobservable systems. The XRE-based PRBT framework is applicable to a large class of passive linear networks satisfying reciprocity. Numerical examples have demonstrated the significant computational savings from the proposed approach over conventional PRBT implementations.

References