An Extension of Cohn's Sensitivity Theorem to Mismatch Analysis of 1-Port Resistor Networks

Sébastien Cliquennois
ST Microelectronics
12, rue Jules Horowitz
38017 GRENOBLE Cedex - France
Contact email: sebastien.cliquennois@st.com

***Abstract*—An analytical expression of statistical mismatch properties of 1-port resistor networks and associated figure-of-merit is proposed, and related to Cohn's sensitivity theorem. This expression is then used to demonstrate matching properties of R-ladders. Experimental verification of this formula is done by comparing theoretical results to Monte-Carlo simulations of random R-networks up to 10 resistors, which are generated by a new graph-based algorithm. Further analysis is performed on this figure-of-merit for all generated networks, leading to more insights into matching properties of R-networks.**

I. Introduction

Integrated circuits (IC) analog designs often use resistive string ladders in conversion circuits (R-2R DACs [1], Flash ADCs [2], or simply in voltage dividers/multipliers [3]). Matching of such structures is generally important for overall performance of design, and a great care is taken in execution of layout in order to limit influence of systematic variations [5]. While synthesis [9], [10] and extraction of electrical properties of such resistive networks is a long-solved problem [6], [7], exact and general mismatch analysis of these networks does not seem to be addressed in a generic formulation. Yet, it represents a relevant selection factor between R-networks [4].

Local mismatches due to random variations of IC processes have been studied for MOSFET [8] and derived for resistors. They are generally characterized for a specific resistor as a random Gaussian distribution of the resistance variation, with an average value R and standard deviation available in a normalized form $\sigma\left[\frac{\Delta R}{R}\right]$.

For any 1-port resistive network (R-network) comprising N resistors $R_i, i \in [1, N]$, and for which equivalent resistance is R_{eq}, it is interesting to derive an analytical expression of $\sigma\left[\frac{\Delta R_{eq}}{R_{eq}}\right]$ such as

$$\sigma\left[\frac{\Delta R_{eq}}{R_{eq}}\right] = f\left(\sigma\left[\frac{\Delta R_1}{R_1}\right], \cdots, \sigma\left[\frac{\Delta R_N}{R_N}\right]\right) \quad (1)$$

where $\sigma\left[\frac{\Delta R_i}{R_i}\right]$ is the normalized standard deviation of each resistor in the network.

Value of $\sigma\left[\frac{\Delta R_{eq}}{R_{eq}}\right]$ immediately assesses the matching quality of a resistor network: the lower the figure, the less spread on resistance final value.

The topology of the R-network has an important impact on this figure : for example, Fig. 1 shows two possible implementations of $R_{eq} = \frac{5}{6}R$ using unit-R resistors. Monte-Carlo simulation of these networks show that for network in Fig. 1a, $\sigma\left[\frac{\Delta R_{eq}}{R_{eq}}\right] \approx 0.836\,\sigma\left[\frac{\Delta R}{R}\right]$, while for network in Fig. 1b, $\sigma\left[\frac{\Delta R_{eq}}{R_{eq}}\right] \approx 0.483\,\sigma\left[\frac{\Delta R}{R}\right]$. Even though network in Fig. 1b uses less resistors (5 instead of 6), its intrinsic mismatch is ≈ 1.7 times better than network in Fig. 1a. This reflects directly in silicon area: in order to obtain the same mismatch value as network in Fig. 1b using network shown in Fig. 1a, the area of unit resistor needs to be increased by a factor of $1.7^2 = 2.89$ because mismatch of resistors varies as $\propto \frac{1}{\sqrt{WL}}$ [8] where W and L are physical dimensions of resistor.

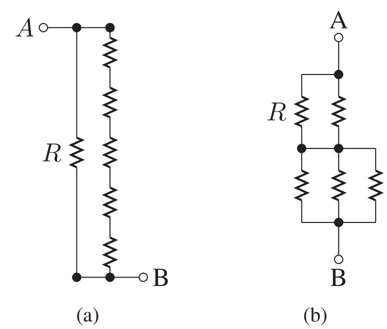


Fig. 1: Two implementations of $\frac{5}{6}R$

While statistical electrical simulations lead to approximations of (1), this paper focuses on providing an exact expression. In section II, derivation of statistical mismatch properties of R-networks is performed, a new figure of merit σ_f introduced and an analytical expression for (1) obtained, using Cohn's sensitivity theorem [6]. Expression is then used to demonstrate properties of R-ladders. In order to experimentally validate this result, a new method for extensive generation 1-port unit-R-networks is proposed in Section III, improving by a factor of 2.7 the reported number of solutions in previous works [11], while proving accuracy of formula compared to Monte-Carlo simulations. Finally, in section IV, proposed σ_f figure is extracted and analyzed for all generated R-networks, highlighting interest of approach.

978-3-9815370-8-6/17/$31.00

II. Mismatch equation derivation

A. General formulation of R_{eq} mismatch

For a 1-port network comprising N resistors $R_1, \cdots, R_N$, the total variation ΔR_{eq} of equivalent resistance R_{eq} can be expressed as

$$\Delta R_{eq} = \frac{\partial R_{eq}}{\partial R_1} \Delta R_1 + \cdots + \frac{\partial R_{eq}}{\partial R_N} \Delta R_N \tag{2}$$

When considering local mismatches, which are mainly due to uncorrelated process variations, all ΔR_i are assumed to be independent random variables, implying on a statistical perspective that $\forall i \neq j, \mathrm{Cov}[R_i, R_j] = 0$. Variance of ΔR_{eq} can then be expressed as:

$$\mathrm{Var}\left[\Delta R_{eq}\right] = \sum_{i=1}^{N} \mathrm{Var}\left[\frac{\partial R_{eq}}{\partial R_i} \Delta R_i\right] \tag{3}$$

Recalling that $\mathrm{Var}[aX] = a^2 \mathrm{Var}[X]$ for a real number a and a random variable X, $\mathrm{Var}\left[\Delta R\right]$ is hence equal to $R^2 \mathrm{Var}\left[\frac{\Delta R}{R}\right]$. Given that $\sigma(X) = \sqrt{\mathrm{Var}(X)}$, (2) can be re-written as

$$\sigma\left[\frac{\Delta R_{eq}}{R_{eq}}\right] = \frac{\sqrt{\sum_{i=1}^{N} {R_i}^2 \left(\frac{\partial R_{eq}}{\partial R_i}\right)^2 \sigma^2 \left[\frac{\Delta R_i}{R_i}\right]}}{R_{eq}} \tag{4}$$

Equation (4) is hence a general solution to (1).

B. Application to unit-R networks

In order to obtain best possible matching, integrated R-networks are generally made of identical unit resistors R sharing same geometrical parameters, hence same $\sigma\left[\frac{\Delta R}{R}\right]$. For such networks, (4) can be simplified to

$$\begin{aligned}\sigma\left[\frac{\Delta R_{eq}}{R_{eq}}\right] &= \frac{R}{R_{eq}} \sqrt{\sum_{i=1}^{N} \left(\frac{\partial R_{eq}}{\partial R_i}\right)^2} \, \sigma\left[\frac{\Delta R}{R}\right] \\ &= \sigma_f \, \sigma\left[\frac{\Delta R}{R}\right]\end{aligned} \tag{5}$$

Equation (5) introduces term $\sigma_f \in]0,1]$ which we shall call 'sigma factor'. It expresses the reduction factor of local mismatch in a R-network compared to unit-R mismatch. The smallest σ_f, the less spread on R_{eq}. Note that σ_f is a topological property of a given R-network.

In order to simplify calculation of (5), Cohn's sensitivity theorem [6] can be used. It states that, for a resistive network,

$$\frac{\partial R_{eq}}{\partial R_i} = \left(\frac{i_{R_i}}{i_{eq}}\right)^2 \tag{6}$$

where i_{eq} is the total current through network, making σ_f calculus direct when R_{eq} and all currents i_{R_i} flowing through unit resistors R_i are known:

$$\sigma_f = \frac{R}{R_{eq}} \sqrt{\sum_{i=1}^{N} \left(\frac{i_{R_i}}{i_{eq}}\right)^4} \tag{7}$$

Equation (7) can be used for hand-calculation of matching quality of R-networks. It can be seen as an extension of Cohn's sensitivity theorem [6] to matching analysis of 1-port resistive networks.

C. Matching of R ladders

A classical usage of 1-port R-networks is the resistor string ladder, shown on Fig. 2, which can be used either in an amplifying or attenuating configuration. Given a reference voltage V_{REF}, a string ladder made of $M+1$ R-networks $\mathcal{R}_i$ can generate M outputs voltages V_i, such as $\forall i \in [1, M], V_k = f_k V_{\text{REF}}$ where f_k are the target attenuating or amplifying factors. Where V_{REF} is applied and where V_i are measured actually depends on amplifying or attenuating configuration.

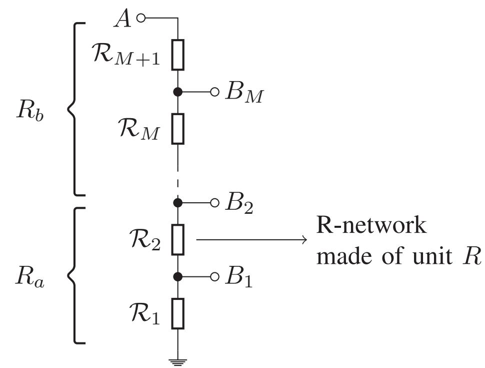


Fig. 2: Resistor Ladder

For this circuit, $\sigma_{f_i}, i \in [1, M+1]$ of R-networks $\mathcal{R}_i$ can be calculated, but the most relevant mismatch figures to obtain are the $\sigma\left[\frac{\Delta f_k}{f_k}\right]$ for $k \in [1, M]$ which describe the dispersion of the attenuating/amplifying factors of structure, for which calculus follows:

1) Attenuating case: In the attenuating configuration case, a reference voltage is set on node A and V_i are measured on nodes B_i. Attenuation factors f_k are hence given by, $\forall k \in [1, M]$

$$f_k = \frac{R_a}{R_a + R_b} \tag{8}$$

with $R_a = \sum_{i=1}^{k} \mathcal{R}_i$ and $R_b = \sum_{i=k+1}^{M+1} \mathcal{R}_i$

Applying total derivation rule on f_k yields to

$$\Delta f_k = \frac{\partial f_k}{\partial R_a} \Delta R_a + \frac{\partial f_k}{\partial R_b} \Delta R_b \tag{9}$$

which after calculus and terms arrangement gives an expression of

$$\frac{\Delta f_k}{f_k} = \frac{R_b}{R_a + R_b} \left(\frac{\Delta R_a}{R_a} - \frac{\Delta R_b}{R_b}\right) \tag{10}$$

which yields to an expression of $\sigma\left[\frac{\Delta f_k}{f_k}\right]$:

$$\sigma\left[\frac{\Delta f_k}{f_k}\right] = \frac{R_b}{R_a + R_b} \sqrt{\sigma^2\left[\frac{\Delta R_a}{R_a}\right] + \sigma^2\left[\frac{\Delta R_b}{R_b}\right]} \tag{11}$$

Resistance R_a is made of k R-networks in series while R_b is made of $M+1-k$ R-networks in series. Because all local variations are uncorrelated, $\mathrm{Var}[\Delta R_a] = \sum_{i=1}^{k} \mathrm{Var}[\Delta \mathcal{R}_i]$. Using (5), one shows that

$$\sigma^2\left[\frac{\Delta R_a}{R_a}\right] = \frac{1}{{R_a}^2}\sum_{i=1}^{k}\left(\mathcal{R}_i\sigma_{f_i}\right)^2\sigma^2\left[\frac{\Delta R}{R}\right] \tag{12}$$

$$\sigma^2\left[\frac{\Delta R_b}{R_b}\right] = \frac{1}{{R_b}^2}\sum_{i=k+1}^{M+1}\left(\mathcal{R}_i\sigma_{f_i}\right)^2\sigma^2\left[\frac{\Delta R}{R}\right] \tag{13}$$

Inserting (12) and (13) in (11) and simplifying leads to the desired expression of $\sigma_{f,k}$ such as $\sigma\left[\frac{\Delta f_k}{f_k}\right] = \sigma_{f,k}\sigma\left[\frac{\Delta R}{R}\right]$ for attenuating structures:

$$\sigma_{f,k} = f_k\frac{R_b}{R_a}\sqrt{\sum_{i=1}^{k}\left(\frac{\mathcal{R}_i\sigma_{f_i}}{R_a}\right)^2 + \sum_{i=k+1}^{M+1}\left(\frac{\mathcal{R}_i\sigma_{f_i}}{R_b}\right)^2} \tag{14}$$

which can be easily calculated by sole knowledge of resistance values and currents, applying (7) to calculate σ_{f_i} for each $\mathcal{R}_i$.

2) Amplifying case: A similar calculus can be performed for amplifying structures. In this case, a reference voltage is set on selected node B_i and V_i is measured on node A where an arbitrary current can flow. In this configuration, $\forall k \in [1, M]$

$$f_k = 1 + \frac{R_b}{R_a} \tag{15}$$

Using (9) expression of $\frac{\Delta f_k}{f_k}$ is now

$$\frac{\Delta f_k}{f_k} = \frac{R_b}{R_a+R_b}\left(\frac{\Delta R_b}{R_b} - \frac{\Delta R_a}{R_a}\right) \tag{16}$$

which is opposite of (10). Because $\mathrm{Var}(X) = \mathrm{Var}(-X)$, expression of $\sigma_{f,k}$ for amplifying structures is actually identical to (14) calculated for attenuating structures

III. R-networks Generation

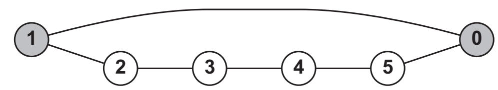
In order to experimentally verify (7) and to create a reference database of 1-port unit-R-networks (including figure-of-merit σ_f) - which can be interesting for further synthesis of R-networks - an extensive generation of 1-port resistive networks comprising from $N = 1$ to $N = 10$ resistors is performed.

This R-network generation problem has been addressed in [11], [12], but only for series and parallel networks, and not including mismatch study.

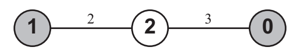
A. Graph representation of R-networks

Conductance graph is a classical representation of resistor networks [13]. The two unit-R networks in Fig. 1 can be represented with the two undirected weighted conductance graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ shown in Fig. 3.

Because networks are made of unit-R - in the rest of analysis, $R = 1$ will be assumed - the weight of graph edges, representing normalized conductance, are integers. When conductance is unity, edge label is omitted for readability. Graph vertices are labeled with integers, which is needed for (Modified) Nodal Analysis [14] later used to extract electrical properties of corresponding network.



(a) Graph $\mathcal{G}_1$ for Fig. 1a



(b) Graph $\mathcal{G}_2$ for Fig. 1b

Fig. 3: Conductance Graphs examples

Vertices ⓪ and ① are the ports of the network. Port ⓪ is grounded, while a current can be injected in ① in order to calculate network R_{eq} and currents through all resistors.

B. Electrical analysis of R-networks

Simple nodal analysis [14] is sufficient to obtain values of voltages at every vertex of graph. The conductance matrix G of graph is obtained by computing its Laplacian matrix, and removing line and columns corresponding to vertex ⓪. As an example, G matrix for graph $\mathcal{G}_2$ is

$$G = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

In order to obtain resistance value and node voltages, a current of $1\,A$ is pushed in vertex ①, resulting in the following $GV = I$ formulation:

$$\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{17}$$

where v_1 and v_2 are voltages at nodes ① and ②. Linear system of equations (17) is simply solved in Scilab [15] using backslash operator, leading to solution

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0.8333 \\ 0.3333 \end{bmatrix} \tag{18}$$

Note that because injected current $i = 1\,A$ and $R_{eq} = v_1/i$, then normalized value of R_{eq} is equal to value of v_1 (here 0.8333).

Currents flowing through resistors are obtained using Ohm's law on every graph edge: $i_{a-b} = G'(v_a - v_b)$, which can also be calculated in matrix form:

$$\begin{bmatrix} i_{1-2} \\ i_{2-0} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{19}$$

Here, $[i_{1-2}\ i_{2-0}]^T = [0.5\ 0.333]^T$, which returns current i_{1-2} flowing trough *each* resistor connected between nodes ① and ②.

As a summary, from conductance graph representation of any R-network, two matrices G and G' are generated, equation $GV = I$ with $I = [1\,0\,\ldots\,0]^T$ is solved for V to obtain voltage values at graph vertices (and R_{eq}), then $I_{a-b} = G'V$ returns all currents in network resistors. Finally, these values are injected into (7) - with $R = 1$ and $i_{eq} = 1$ - to obtain σ_f.

As an illustration, σ_f for graph $\mathcal{G}_2$ is calculated using (19) , (18) and (7):

$$\sigma_f = \frac{1}{0.833}\sqrt{2 \times 0.5^4 + 3 \times 0.333^4} = 0.483$$

C. Generation of 1-port R-networks

1) Problem statement: Counting of connected graphs is a subject which is addressed in graph theory [16], and in the general case, this figure quickly grows with the number of vertices - e.g. 26704 solutions for $n = 6$). However, in order to generate R-networks, we limit the weighted conductance connected graph generation thanks to following constraints:

- Graph has a minimum of two vertices ⓪ and ①
- Vertex degree must be ≥ 2, except for ⓪ and ①
- Two equivalent graphs have same k, σ_f and R_{eq}

Where k is sum of edges weight, i.e. number of resistors.

If generation is limited to N unit resistors, $N+1$ will be the largest possible number of graph vertices, corresponding to N unit resistors in series between ① and ⓪.

The proposed algorithm is recursive: starting from a graph $G_{k,j}$ with k resistors and j vertices, generate graphs with $k+1$ resistors which verify the above-mentioned constraints.

2) Proposed algorithm: Decompose the generation algorithm in 5 sub-steps, starting from $G_{k,j}$:

- Generate new graphs with $j+1$ vertices, adding a new (unit) edge and vertex after each existing edge (or before first/last vertex). This will create N_e+2 new graphs where N_e is number of edges in graph.
- Generate new graphs by inserting a new resistor between two existing nodes. This will create C_2^j new graphs, C_2^j being number of combinations of 2 vertices among j.
- Generate new graphs by disconnecting tail of one edge with weight ≥ 2 from a vertex, and reconnect this "free edge" tail to every other vertex using an extra resistor. It will generate $N-1$ graphs for each edge of weight ≥ 2.
- Search for all equivalent networks (same R_{eq} and σ_f) in the generated set and keep only one.
- Remove solutions which include resistors in which no current flows.

This algorithm guarantees that no dangling vertex will be created, because every vertex creation or edge removal operation is designed such as internal vertex degree remains ≥ 2.

3) Example application: Starting from graph $\mathcal{G}_2$, algorithm is run and generates a total of 9 new graphs before removal of equivalent networks.

The 3 first graphs shown Fig. 4 in are obtained by inserting a new resistor after/before existing vertices.

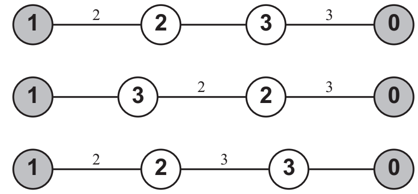


Fig. 4: Insertion of a new vertex and new edge

Figure 5 shows addition of one resistor on existing vertices.

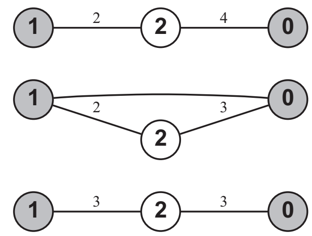


Fig. 5: Extra resistor, existing vertices

Finally, the "free edge" method results in the 3 graphs shown in Fig. 6.

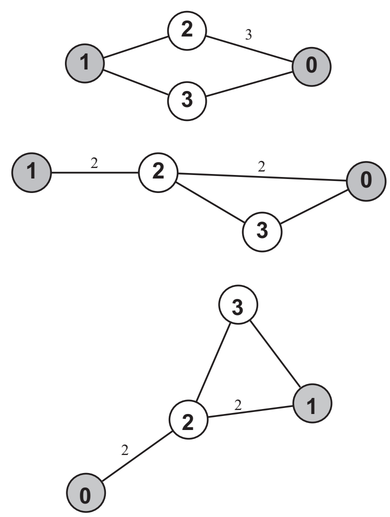


Fig. 6: "free edge" connected to new vertex 3

D. R-networks from $N = 1$ to $N = 10$

Algorithm described in section III-C2 is applied to the seed network $N = 1$ (i.e. 1 resistor connected between vertices ① and ⓪), which generates 2 graphs for $N = 2$, and operation is recursively repeated on all generated networks until $N = 10$.

Table I summarizes the results, compared to the number of generated networks reported in [11] where only serial and parallel configurations are used.

All solutions were computed with Scilab [15] on a laptop PC, the slowest computation (for $N = 10$) taking less than one minute. Almost ten thousands (9863) unique solutions are generated, which is 2.7 times more than reported in [12],

N	graph count	CPU (s)	graph count [11]	min σ_f	max σ_f
1	1		1	1.0	1.0
2	2	0.01	2	0.707	0.707
3	4	0.01	4	0.577	0.707
4	9	0.01	9	0.5	0.763
5	22	0.04	22	0.447	0.806
6	60	0.14	53	0.408	0.837
7	176	0.53	131	0.378	0.859
8	564	2.06	337	0.352	0.876
9	1932	8.75	869	0.333	0.89
10	7094	56.1	2213	0.316	0.9
	9863		3641		

TABLE I: Summary of generated R-networks

[11]. The generated networks are then stored in a database for further analysis and usage in resistor ladder synthesis algorithms.

IV. Mismatch figures of generated networks

A. Experimental verification of σ_f

Definition of σ_f gives an insight on local matching property of resistor networks. In this section, an experimental verification of (7) is performed using Monte-Carlo simulations, implemented in Scilab.

In order to perform these simulations, the algorithm which creates matrix G from graph representation is slighted modified to generate a symbolic matrix. For network in Fig. 3b, generated symbolic G matrix is

$$G = \begin{bmatrix} g(1)+g(2) & -g(1)-g(2) \\ -g(1)-g(2) & g(1)+g(2)+g(3)+g(4)+g(5) \end{bmatrix}$$

A vector g is generated using a Gaussian random number generator implemented using Scilab function `grand`, with specified mean and standard deviation, allowing to generate uncorrelated random numbers. Matrix G and σ_f are then evaluated using (7). The process is repeated 3000 times, and values of R_{eq} and σ_f stored for each evaluation.

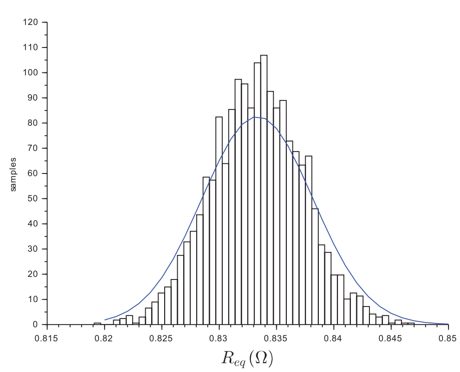


Fig. 7: Histogram and calculated PDF of R_{eq}

Figure 7 shows a normalized histogram of simulated R_{eq} for network in Fig. 1b, together with calculated Gaussian Probability Density Function. The theoretical σ_f value is $4.83 \times 10^{-3}\,\Omega$ while a simulation on 3000 samples yields a standard deviation (using `stdev`) of $4.835 \times 10^{-3}\,\Omega$ showing an error of 0.08%.

In order to confirm the results on other networks, Monte-Carlo simulations have been run on random networks among the generated database of R-networks.

test#	R	$\sigma_f \times 10e^{-3}$ (calculated)	$\sigma_f \times 10e^{-3}$ (simulated)	error (%)
1	0.454	6.441	6.347	-1.47
2	0.769	5.658	5.690	0.57
3	1.333	4.018	4.007	-0.28
4	0.647	6.795	6.782	-0.20
5	0.692	7.190	7.077	-1.58
6	0.429	5.151	5.112	-0.76
7	2.8	5.551	5.610	1.06
8	0.428	5.151	5.148	-0.06
9	0.786	7.927	7.995	0.86
10	5.5	4.116	4.094	-0.54

TABLE II: σ_f for 10 random networks

Table II gathers some of the results, which show that calculated σ_f is actually relevant, error between theoretical and simulated on its value being always less that 1.6%.

B. Analysis of generated networks

Almost ten thousand 1-port resistor networks have been generated, R_{eq} and σ_f calculated for all of them. Figures 8 and 9 (which is a zoom on $R_{eq} \in [0, 2]$) are scatter plots of R_{eq} vs. σ_f.

Analysis of the plots show some expected and some less expected results. Both sparsity of solutions for $R_{eq} \geq 2$ and decreasing of σ_f values are expected. For bigger values of R_{eq} there are less and less possibilities using a maximum of 10 resistors (e.g. only 1 solution for $10R$, $9R$ and $8.5R$). Decreasing σ_f is explained by the fact that for bigger R_{eq}, factor $1/R_{eq}$ in (7) becomes important.

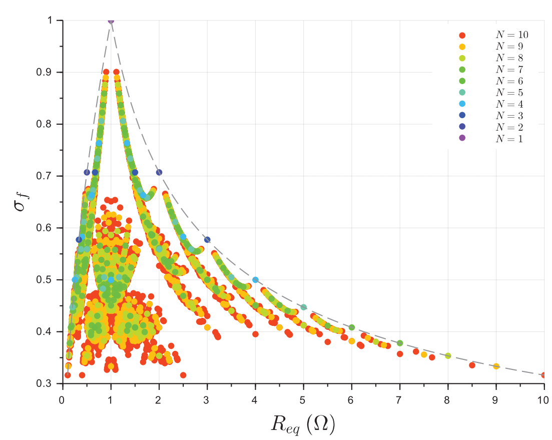


Fig. 8: Scatter plot of σ_f vs. R_{eq} for $N \in [1, 10]$

Some other phenomena would require further analysis: an envelope of maximum σ_f can be seen, with spikes of envelope being the iR and R/i values where $i \in \mathbb{N}^*$.

Zooming on R_{eq} range from 1/10 to 2.0 in Fig. 9, several remarks can be done:

- For integer values $i \geq 1$ of R_{eq}. σ_f is equal to $1/\sqrt{i}$ while for fractional values $1/i$, $\sigma_f = \sqrt{i}$, as illustrated by dotted gray lines on Figs 8 and 9.
- For some ranges (e.g. $R_{eq} \in [0.6, 0.8]$), there is at least a factor 2 between min and max σ_f. This remark justifies importance of σ_f. It worth exploring different solutions for R-networks because there can be a significant difference in matching properties.
- Density of solutions is important in the $[1/N, 2]$ region, which has a direct consequence for synthesis algorithms: when trying to generate a resistance $R_0 \gg R$ using unit resistors R, immediate approach is to generate the integer part $\lfloor R_0/R \rfloor$ as series resistors, and synthesize the fractional part $\{R_0/R\}$ which lies in $[1/N, 1[$. Figure 9 shows generating $\lfloor R_0/R \rfloor - 1$ as series resistors, then looking for remaining part in $[1/N, 2[$ may also be a second interesting option, thanks to important density of "low σ_f" solution in $[1, 2]$ region.

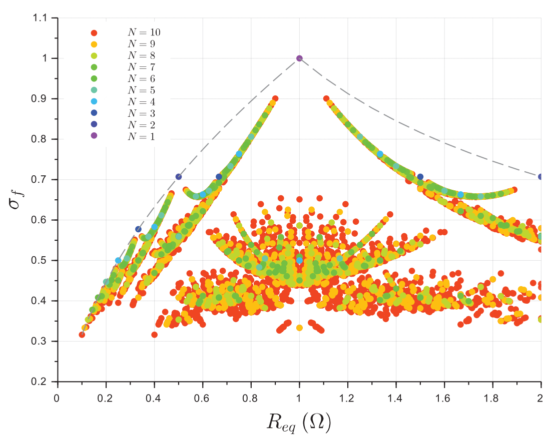


Fig. 9: σ_f vs. R_{eq} for $N \in [1, 10]$ with $R_{eq} < 2$

V. Conclusion

Local mismatch characteristics of matched resistor arrays are an important parameter for a class of IC analog blocks such as voltage dividers, amplifiers, or ladder DACs and ADCs. This paper introduced σ_f, a factor-of-merit of 1-port R-networks, allowing direct calculation of their local statistical matching properties. An analytical expression for σ_f is derived, which can been seen as an extension of Cohn's sensitivity theorem for matching. Expression of attenuating/amplifying factor matching is also derived for generic resistor string ladder structures. An experimental validation of σ_f is achieved by generating 1-port unit-R networks using a new graph-based algorithm which creates 2.7 times more solutions than previously reported series/parallel generation schemes. The generated networks are then simulated in Monte-Carlo approach to validate the closed-form expression of their standard deviation. Finally, analysis of σ_f of all generated R-networks gives more insights into synthesis of R-networks. Further work will consist in studying matching-aware R-networks synthesis algorithms which can benefit both from the extended generated database of solution generated for $N \leq 10$ and from insights in matching properties of generated solutions.

References

[1] M. Kennedy, "On the robustness of r-2r ladder dacs," *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, vol. 47, no. 2, pp. 109–116, Feb 2000.

[2] K. Uyttenhove and M. Steyaert, "A 1.8-v 6-bit 1.3-ghz flash adc in 0.25-μ m cmos," *Solid-State Circuits, IEEE Journal of*, vol. 38, no. 7, pp. 1115–1122, July 2003.

[3] S. Tsao, "A 25-bit reference resistive voltage divider," *Instrumentation and Measurement, IEEE Transactions on*, vol. IM-36, no. 2, pp. 285–290, June 1987.

[4] C.-W. Kok and W.-S. Tam, *CMOS voltage references: an analytical and practical perspective*. Hoboken, NJ: Wiley-IEEE Press, 2013.

[5] R. Hastings, *The Art of Analog Layout*. Prentice Hall, 2001.

[6] R. Cohn, "The resistance of an electrical network," *Proceedings of the American Mathematical Society*, vol. 1, no. 3, pp. 316–324, 1950.

[7] L. O. Chua and P. Y. Lin, *Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques*. Prentice Hall Professional Technical Reference, 1975.

[8] M. J. Pelgrom, A. C. Duinmaijer, A. P. Welbers, *et al.*, "Matching properties of mos transistors," *IEEE Journal of solid-state circuits*, vol. 24, no. 5, pp. 1433–1439, 1989.

[9] M. Z. Q. Chen, K. Wang, M. Yin, C. Li, Z. Zuo, and G. Chen, "Synthesis of n-port resistive networks containing 2n terminals," *International Journal of Circuit Theory and Applications*, vol. 43, no. 4, pp. 427–437, 2013.

[10] F. Boesch, "Some aspects of the analysis and synthesis of resistor networks," *Network and Switching theory: A NATO Advanced study institute*, p. 189, 1968.

[11] S. A. Khan, "The bounds of the set of equivalent resistances of n equal resistors combined in series and in parallel," *arXiv preprint arXiv:1004.3346*, 2010.

[12] A. Amengual, "The intriguing properties of the equivalent resistances of n equal resistors combined in series and in parallel," *American Journal of Physics*, vol. 68, no. 2, pp. 175–179, 2000.

[13] S. Seshu and M. B. Reed, "Linear graphs and electrical networks," 1961.

[14] C.-W. Ho, A. E. Ruehli, and P. A. Brennan, "The modified nodal approach to network analysis," *Circuits and Systems, IEEE Transactions on*, vol. 22, no. 6, pp. 504–509, Jun 1975.

[15] Scilab Enterprises, *Scilab: Free and Open Source software for numerical computation*, Scilab Enterprises, Orsay, France, 2012.

[16] B. Bollobás, "The asymptotic number of unlabelled regular graphs," *J. London Math. Soc*, vol. 26, no. 2, pp. 201–206, 1982.