Implicit index-aware model order reduction for RLC/RC networks

Nicodemus Banagaaya*, Giuseppe Ali†, Wil. H. A. Schilders* and Caren Tischendorf‡
*Dept. of Mathematics and Computer Science
Eindhoven University of Technology, The Netherlands
Email: n.banagaaya@tue.nl, w.h.a.schilders@tue.nl
†Dept. of Physics, University of Calabria and INFN,
Gruppo collegato di Cosenza, Arcavacata di Rende, I-87036 Cosenza, Italy.
Email: giuseppe.ali@unical.it
‡Institute of Mathematics, Humboldt-Universität zu Berlin, 10099 Berlin, Germany.
Email: caren.tischendorf@math.hu-berlin.de

Abstract—This paper introduces the implicit-IMOR method for differential algebraic equations. This method is a modification of the Index-aware model order reduction (IMOR) method proposed in our earlier papers which is the explicit-IMOR method. It also involves first splitting the differential-algebraic equations (DAEs) into differential and algebraic parts using a basis of projectors. In contrast with the explicit-IMOR method, the implicit-IMOR method leads to implicit differential and algebraic parts. We demonstrate the implicit-IMOR method using the RLC/RC networks, but it can also be applied to other problems which lead to differential-algebraic equations.

I. INTRODUCTION

Consider a linear RLC electric network, that is, a network which connects linear capacitors, inductors and resistors, and independent current sources \( e(t) \in \mathbb{R}^n \). The unknowns which describe the network are the node potentials \( e(t) \in \mathbb{R}^n \), and the currents through inductors \( J_L(t) \in \mathbb{R}^n \). Following the formalism of Modified Nodal Analysis (MNA) [1], we introduce: the incidence matrices \( A_C \in \mathbb{R}^{n_c,n_c} \), \( A_L \in \mathbb{R}^{n_L,n_L} \) and \( A_R \in \mathbb{R}^{n_R,n_R} \), which describe the branch-node relationships for capacitors, inductors and resistors; the incidence matrix \( A_I \in \mathbb{R}^{n_I,n_I} \), which describe this relationship for currents sources. Then this leads to a descriptor system for the unknown \( x = (e, J_L) \) given by

\[
\begin{bmatrix}
A_C & A_C^T & 0 \\
0 & L & A_L \\
E & 0 & A_I
\end{bmatrix}
\begin{bmatrix}
dx \\
dJ_L \\
x + A_I x_I
\end{bmatrix} = \begin{bmatrix}
-A_R G A_R^T & -A_L & 0 \\
A_R G A_R^T & -A_L & 0 \\
0 & 0 & -A_I
\end{bmatrix} x + \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} z,
\]

(1)

with consistent initial data

\[
x(t_0) = x_0.
\]

(2)

Here, \( C \in \mathbb{R}^{n_c,n_c} \), \( L \in \mathbb{R}^{n_L,n_L} \) and \( G \in \mathbb{R}^{n_c,n_R} \) are the capacitance, inductance and conductance matrices, which are assumed to be symmetric and positive-definite. Note that we consider a network of only current sources for simplicity but also voltages sources can be used. If \( E \) is singular, (1) is a differential algebraic equation (DAE) otherwise it is an ordinary differential equation (ODE). In this paper we assume that \( E \) is singular, thus we are considering DAEs. The dimension of the DAE system (1) is \( N = n + n_L \).
achieve system approximation by moment matching. Among these methods are PRIMA [7], the structure preserving version SPRIM [8] and so on. The most important advantages offered by PRIMA are: the applicability to MIMO systems and passivity preservation. However the two main limitations of PRIMA are that it does not preserve the MNA structure of the original system and the index of the system, i.e., It leads to ODE reduced-order models. Moreover, PRIMA leads to wrong reduced-order models for DAEs of higher index [2]. The problem of not preserving the MNA structure was solved by its structure preserving version SPRIM. However, index problem was not solved till now. In [3] and [2] we proposed a index-aware model order reduction (IMOR) method which preserves the index of the system and can be used to even reduce higher index DAEs. In this method we first split the DAE into differential and algebraic parts using projectors and their respective bases. Then we can use conventional MOR methods such as PRIMA, to reduce the differential part and then develop techniques to reduce the algebraic parts. However the explicit-IMOR method proposed in [3], [2] involves matrix inversion which may be computationally expensive. In this paper we modify the method by splitting the DAEs without matrix inversions, which we call the implicit-IMOR (IIlOM) method. This paper is organized as follows: Sect. II, we briefly discuss about the PRIMA method, then in Sect. III, we discuss about the decoupling of the RLC and RC networks without matrix inversions. Then in Sect. IV, we discuss the IMOR method. Finally we carry out experiments using DAEs from electric power grid models and then the conclusions.

II. MODEL ORDER REDUCTION

At the heart of model order reduction lies the desire to approximate the behavior of a large dynamical system in an efficient manner, so that the resulting approximation error is small [5]. Other requirements are: the preservation of important system properties, of its physical interpretation, and an efficient implementation. In other words, the reduced-order model must be computationally cheaper to solve than its original model. We replace the original system (4) for \( x \in \mathbb{R}^n \), with output \( y \in \mathbb{R}^r \), with the reduced system (6) for \( x_r \in \mathbb{R}^r \), with output \( y_r \in \mathbb{R}^s \). The unifying approach for obtaining a reduced-order model from an original system is via a Petrov-Galerkin projection: \( E_r = W^T E V \), \( A_r = W^T A V \), \( B_r = W^T B \) and \( C_r = V^T C \), where \( V, W \in \mathbb{R}^{n,rm} \) are the matrices whose \( r \ll n \) columns form bases for the relevant subspaces pertaining to the reduction method chosen. Model Order Reduction methods differ in the way the decomposition is performed, this in turn dictates how the projection matrices \( V \) and \( W \) are constructed. There are many MOR methods, but in this paper we shall focus on PRIMA [7] which is the most popular reduction method for electric circuits. In this method, one assumes the Galerkin projection, i.e \( V = W \in \mathbb{R}^{n,rm} \). Then the projection \( V \) is constructed as follows: Choosing arbitrary expansion point \( s_0 \in \mathbb{C} \), then we consider order-\( r \) Krylov subspace generated by \( M = (s_0 E - A)^{-1} E \) and \( R = (s_0 E - A)^{-1} B \), that is \( K_r(M, R) = \text{span}\{R, MR, \cdots, M^{r-1}R\} \), \( r \leq n \) and denoted by \( V \in \mathbb{R}^{n,rm} \) the matrix of an orthonormal basis for \( K_r(M, R) \) so that \( V^T V = I \). However the PRIMA method is not valid for DAEs of higher index and does not preserve the index of the DAE system. In [3] and [2], they proposed a new MOR method for index-1 and -2 DAEs respectively, which they called the IMOR method. This method involves first splitting the DAE into differential and algebraic parts. Then one can apply any model order reduction method on the differential part and also reduce the algebraic parts. In this paper, we call it the explicit-IMOR method. However the explicit-IMOR method involves matrix inversions which may be computationally expensive for very large systems, this motivated us to develop its no inversion version which we call the implicit-IMOR (IMOR) method.

III. DECOUPLING OF RLC/RC NETWORKS

In this section, we introduce the implicit splitting of DAEs using projectors and their corresponding bases. This splitting is different from that proposed in [3], although the approach is almost the same. Here we consider the splitting of index-1 RLC/RC networks but the same procedure can be applied to any index-1 system. In order to decouple system (1) we need to first construct the matrix and projector chains of the matrix pencil \((E, A)\) using the definition of tractability index as defined in [11]. Setting \( E_0 := E, A_0 := A \), further

\[
E_{j+1} = E_j - A_j Q_j, \quad A_{j+1} := A_j P_j, \quad j \geq 0, \quad (7)
\]

whereby \( Q_j \) denotes a projector onto the nullspace \( \text{Ker}\) \( E_j \) and its complementary projector \( P_j = I - Q_j \). The sequence \( E_0, E_1, \cdots \), is known to become stationary, i.e., \( E_{j+1} = E_j, j \geq 0 \), where \( \mu \) is the tractability index, supposing the matrix pencil \( LE = A \) is regular.

Assuming the matrix pencil \((E, A)\) is regular, we can compute the tractability index of (4a). This can be done as follows:

We first set \( E_0 = E, A_0 = A \):

\[
E_0 = \begin{pmatrix} A_C C_C^T & 0 \\ 0 & L \end{pmatrix}, \quad A_0 = \begin{pmatrix} -A_R G_A^T & -A_L \\ A_L & 0 \end{pmatrix}.
\]

We then choose a projector \( Q_0 \) such that \( \text{Im} \ Q_0 = \text{Ker}\ E_0 \) and its complementary projector \( P_0 = I - Q_0 \). For this class of problem, we can just choose a projector \( Q_C \) that projects onto the kernel of \( A_C \), and \( P_C = I - Q_C \). Then we can obtain:

\[
Q_0 = \begin{pmatrix} Q_C & 0 \\ 0 & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} P_C & 0 \\ 0 & I \end{pmatrix}.
\]

Then:

\[
E_1 = E_0 - A_0 Q_0 = \begin{pmatrix} A_C C_C^T + A_R G_A^T Q_C & 0 \\ -A_L Q_C \end{pmatrix},
\]

\[
A_1 = A_0 P_0 = \begin{pmatrix} A_R G_A^T & -A_L \\ A_L P_C & 0 \end{pmatrix}.
\]

It can easily be proved that if, we have the condition:

\[
\ker(A_C, A_R^T) = \{0\}, \quad (8)
\]

then we find that \( x \in \ker\ E_1 \) if and only if \( Q_C e = 0 \) and assuming \( L \) is a nonsingular matrix. Thus the condition (8) is equivalent to the index-1 condition, i.e., \( E_1 \) is non-singular.

If the condition (8) is not satisfied, we need to iterate the procedure. In this case, we need to introduce a projector
Q₁ onto the Ker E₁, satisfying the additional requirements Q₁Q₀ = 0. To do this, we need to first introduce a projector

\[ Q₁ = \begin{pmatrix} Q_{CR} & 0 \\ L^{-1}A₁^TQ₂ & 0 \end{pmatrix} \]  

(9)

which also projects onto the Ker E₁, where Q₁CR is a projector onto ker(A₁, A₁P₁). Then, the projector Q₁ = -Q₁(E₁ - A₁Q₁)⁻¹A₁ is a projector onto the Ker E₁, that satisfies the condition Q₁Q₀ = 0. Thus, we can find E₂ = E₁ - A₁Q₁ and A₂ = A₁P₁. If E₂ is nonsingular, then (4) is an index-2 system. In this paper, we assume that index-1 condition (8) is satisfied thus (4a) is an index-1 system. From this point, we assume E₁ is nonsingular unless otherwise stated.

A. Index-1 RLC network

In this section, we decouple index-1 systems of the form (4a). We first construct bases for projectors Q₁ and P₀ given by q₁ ∈ Rⁿ₁, p₁ ∈ Rⁿ₂, and their respective inverses are given by q₁ᵀ ∈ Rⁿ⁻¹, p₁ᵀ ∈ Rⁿ⁻₂, where n = n₁ + n₂. Thus the bases for projectors Q₀ and P₀ are given by q = (q₁, 0) ∈ Rⁿ₁, p = (p₁, 0, I) ∈ Rⁿ₂. The differential variable ξₚ and the algebraic variable ξₚ are given by

\[ ξ₁ = p₁ᵀx ∈ Rⁿ₂⁻¹ + nₑ, \]

(10)

In [3], they decoupled index-1 systems using the above bases and their corresponding inverses but they also had to compute the inverse of E₁. But inverting E₁ might be computationally expensive and leads to very dense matrices of the decoupled system. In this paper, we try to avoid inverting E₁ as follows: On additional to the above bases, we construct p₁ᵀ ∈ Rⁿ₂⁻¹ + nₑ, q₁ᵀ ∈ Rⁿ⁻¹ such that p₁ᵀA_q = 0 and q₁ᵀE_p = 0, given by p₁ ∈ ker(q₁ᵀA₁RGA₁ᵀ - q₁ᵀA₁L) and q₁ = q₁, since E₁ is symmetric. Without loss of generality, index-1 system (4a) can be decoupled as

\[ \begin{align*}
E_p & = \begin{pmatrix} A_p & B_p \\
B_p & C_p \end{pmatrix}, \\
\hat{p}^T E_p \hat{p} & = \hat{p}^T A_p \hat{p} + \hat{p}^T B_p \hat{u}, \\
\hat{q}^T A_q \hat{q} & = \hat{q}^T A_q \hat{q} + \hat{q}^T B_q \hat{u},
\end{align*} \]  

(11a)

(11b)

where (11a) and (11b) is the differential and algebraic parts. We can observe that there is no inversion of matrices, thus this decouple system is computationally cheaper to derive than it’s counter part in [3]. If we use matrices in (1) the algebraic part (11b) can be written as

\[ q₁ᵀA₁RGA₁ᵀq₁ = -q₁ᵀ(A₁RGA₁ᵀA₁) p₁ - q₁ᵀA₁ \hat{u}. \]

(12)

The output equation (4b) can also be decomposed as

\[ y = C_p^T \hat{p} + C_q^T \hat{q}, \]

(13)

where C_p = pᵀC and C_q = qᵀC. If we assume C = B, thus we have C_p = [-pᵀA₁ 0] and C_q = -qᵀA₁. Example below illustrates how to decouple RLC networks of the form (1) using the above proposed procedure.

Example 1: Consider an RLC circuit network with two current sources as shown in figure below. The incidence ma-

![Two port RLC circuit example](image)

trices for capacitors, resistors, inductors and current sources are given by

\[ A_C = \begin{pmatrix} 0 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0 \end{pmatrix}, \quad A_R = \begin{pmatrix} 1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 0 \end{pmatrix}, \]

\[ A_L = \begin{pmatrix} 0 & 0 & 0 \\
0 & 1 & -1 \\
-1 & 0 & 0 \end{pmatrix}, \quad A_I = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}. \]  

(14)

The capacitance, inductance and and conductance matrices are given by

\[ C = \begin{pmatrix} C₁ & 0 & 0 \\
0 & C₁ & 0 \\
0 & 0 & C₂ \end{pmatrix}, \quad L = L₁ \quad \text{and} \quad \mathcal{G} = \begin{pmatrix} G₁ & 0 & 0 \\
0 & G₂ & 0 \\
0 & 0 & G₃ \end{pmatrix}. \]  

(15)

Substituting (14) and (15) into (1) we obtain a DAE with system matrices given by

\[ \begin{align*}
E & = \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix}, \\
A & = \begin{pmatrix} -G₁ & G₁ & 0 & 0 & 0 \\
G₁ & -G₁ - G₂ & G₂ & 0 & 0 \\
0 & G₂ & -G₂ & 0 & 0 \\
0 & 0 & 0 & G₃ & 1 \\
0 & 1 & 0 & -G₃ & 0 \\
0 & 0 & 1 & -1 & 0 \end{pmatrix}, \\
B & = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad C = B, \quad u = \begin{pmatrix} u₁ \\
u₂ \end{pmatrix}.
\end{align*} \]

(16)

and the unknown x = (e₁, e₂, e₃, e₄, J₁, J₂)ᵀ. This system is solvable since det(λE - A) ≠ 0 and it is of index -1 since the incidence matrices A₁ and A₁ satisfy condition (8). Following the procedure in the previous section we obtain
the projectors \( Q_C \) and \( P_C \) given by
\[
Q_C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]

The bases of \( Q_C, P_C \) and their respective inverses are given by
\[
q_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad p_c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad q_c^T = q_c^T, p_c^T = p_c^T.
\]

Thus the bases of projectors \( Q_0, P_0 \) and their respective inverses are given by
\[
q = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad q^T = q^T, p^T = p^T.
\]

Using (10) the differential and algebraic unknowns are given by
\[
\xi_p = p^T x = \begin{bmatrix} e_2 \\ e_3 \end{bmatrix}, \quad \xi_q = q^T x = \begin{bmatrix} e_1 \\ e_4 \end{bmatrix}.
\]

The decoupling bases \( \tilde{p} \) and \( \tilde{q}^T \) are given by
\[
\tilde{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -\frac{1}{C_3} \end{bmatrix} \quad \text{and} \quad \tilde{q} = q.
\]

Substituting (16) and (17)-(19) into (11), we obtain the decoupled system with coefficient matrices
\[
E_p = \begin{bmatrix} C_1 + C_c & -C_c & 0 \\ -C_c & C_2 + C_c & 0 \\ 0 & 0 & L \end{bmatrix}, \quad A_p = \begin{bmatrix} -G_2 & G_2 & 0 \\ G_2 & -G_2 & -1 \\ 0 & 1 & -\frac{1}{C_3} \end{bmatrix},
\]
\[
B_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad E_q = \begin{bmatrix} G_1 & 0 \\ 0 & G_3 \end{bmatrix}, \quad A_q = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]
\[
B_q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

**B. RC network**

The RC network equation can easily be derived from (1) if we eliminate the the inductors in the RLC network leading to
\[
\frac{\text{d}e}{\text{d}t} = -A_R \mathbf{G} A_R^\top e + -A_I i, \tag{21}
\]

with consistent initial data \( e(0) = e_0 \),

where the node potentials \( e(t) \in \mathbb{R}^n \) are the only unknowns. Here, still \( C \in \mathbb{R}^{n_c,n_c} \), and \( G \in \mathbb{R}^{n_c,n_G} \) are the capacitance and conductance matrices, which are assumed to be symmetric and positive-definite. This system can also be written in the form (4), we can assume \( C = B \). Assuming the matrix pencil \((E, A)\) is regular, then we can also compute its tractability index as follows: Setting \( E_0 = E \) and \( A_0 = A \) leads to
\[
E_0 = A_c C A_c^\top, \quad A_0 = -A_p G A_p^\top, \quad B = -A_I \quad \text{and} \quad u = \mathbf{1}.
\]

We denote by \( Q_0 = Q_c \) the projector onto the kernel of \( A_c^\top \), and set \( P_0 = P_c = I - Q_c \), such that \( P_C Q_C = Q_C P_C = 0 \). Then, we can find
\[
E_1 = E_0 - A_0 Q_0 = A_c C A_c^\top + A_R G A_R^\top Q_C,
\]
\[
A_1 = A_0 P_0 = A_R G A_R^\top P_C.
\]

It is also easy to show that If, we have the conditions:
\[
\text{Ker}(A_C, A_R)^\top = \{0\}, \tag{22}
\]
then we find that \( e \in \ker(E_1) \) if and only if \( Q_c e = 0 \). Thus the condition (22) is equivalent to the index-1 condition \( \det E_1 \neq 0 \) as for the case for RLC networks with only current sources.

1) **Index-1 RC network:** In this case we construct bases for projectors \( Q_0 = Q_C \) and \( P_0 = P_C \) given by \( q = q_c \in \mathbb{R}^{n,n_1} \), \( p = p_c \in \mathbb{R}^{n,n_2} \) and their respective inverses is given by \( q^T = q_c^T \in \mathbb{R}^{n,n_1} \), \( p^T = p_c^T \in \mathbb{R}^{n_2,n} \), where \( n = n_1 + n_2 \). The differential variable \( \xi_p \) and the algebraic variable \( \xi_q \) are given by
\[
\xi_p = p^T x = p_c^T e \in \mathbb{R}^{n_2}, \quad \xi_q = q^T x = q_c^T e \in \mathbb{R}^{n_1}.
\]

In order to decouple the DAE system, we construct \( \hat{p}^T \in \mathbb{R}^{n_2,n} \), \( \hat{q}^T \in \mathbb{R}^{n,n_1} \) such that \( \hat{p}_0^T A q_0 = 0 \) and \( \hat{q}_0^T E p_0 = 0 \), given by \( \hat{p} \in \ker(q_c^T A_R G A_R^\top) \) and \( \hat{q} = q \) since \( E \) is symmetric. From (11) and (13), we can decouple index-1 RC network as
\[
E_p \xi_p' = A_p \xi_p + B_p u, \tag{23a}
\]
\[
E_q \xi_q' = A_q \xi_q + B_q u, \tag{23b}
\]
\[
y = C_p \xi_p + C_q \xi_q, \tag{23c}
\]

where \( E_p = \hat{p}^T A_c C A_c^\top p, \quad A_p = -\hat{p}^T A_R G A_R^\top p, \quad B_p = -\hat{q}^T A_I, \quad E_q = \hat{q}^T A_R G A_R^\top q, \quad A_q = -\hat{q}^T A_R G A_R^\top p, \quad B_q = -\hat{q}^T A_I, \quad \text{and} \quad C_p^\top = -p_c^T A_I, \quad C_q^\top = -p_c^T A_I. \)

and (23b) is the differential and algebraic parts. We note that the \( E_p \) and \( E_q \) must always be non-singular for any index-1 system.

**IV. IMPLICIT-IMOR METHOD**

In this section, we propose the Implicit -index-aware model order reduction (Implicit-IMOR) method which is the modification of the index-aware model order reduction method proposed in [3], [2]. In this paper, we shall call this method the explicit-IMOR method. In the explicit-IMOR method we apply the reduction on the explicit decoupled system while the implicit-IMOR method we apply it on the implicit decoupled system (23). System (23) can be written in the form (4) given by
\[
\dot{E} \xi' = \tilde{A} \xi + \tilde{B} u, \tag{24a}
\]
\[
y = \tilde{C} \xi, \tag{24b}
\]
Decoupled dimension

Dimension $n$

m

Explicit splitting method

$5727$

16

1

where $\tilde{E} = \begin{bmatrix} E_p & 0 \\ 0 & 0 \end{bmatrix}$, $\tilde{A} = \begin{bmatrix} A_p & 0 \\ A_q & -E_q \end{bmatrix} \in \mathbb{R}^{N,N}$,

$\tilde{B} = \begin{bmatrix} B_p \\ B_q \end{bmatrix} \in \mathbb{R}^{N,m}$, $\tilde{C} = \begin{bmatrix} C_p \\ C_q \end{bmatrix} \in \mathbb{R}^{N,l}$ and the projected state space $\xi = (\xi^T_q^T \xi_q^T) \in \mathbb{R}^N$, where $\xi_p \in \mathbb{R}^{n_p}$, $\xi_q \in \mathbb{R}^{n_q}$ and $N = n_p + n_q$. We note it can easily be proved that system (24) and (4) are equivalent for index-1 systems. Moreover it can be proved that the finite spectrum of the matrix pencil $(E, A)$ is equal to the spectrum of $(E_q, A_p)$. Thus the decoupling procedure preserves the spectrum of the original system. Then from (29), we can observe that the algebraic variable $\xi_q$ satisfies properties suggested in [7]. We note the projectors and their corresponding bases are numerically feasible and can be computed using the LU based routine proposed in [9] for the case of sparse matrices. But for the case of dense matrices SVD based routines have to be used.

V. Numerical experiments

In Tab I, we decouple Electric Power Grids models. These are real world index-1 DAE models which can be downloaded from [12]. In Tab I, $n_p$ and $n_q$ represents the number of differential and algebraic equations, respectively. We can observe that the above reduction of the differential part induces a reduction in the algebraic part (27a) but its dimension is unchanged given by

$$E_q \xi_q = A_q V_p \xi_p + B_q u.$$  \hfill (29)

We have already seen that the differential variable $\xi_p$ is confined to the subspace $V_p = \mathcal{K}_r(M_p, R_p)$ spanned by $V_p$. Then from (29), we can observe that the algebraic variable $\xi_q$ belongs to the subspace $V_q := \text{span}(E_q^{-1} V_q)$ in $\mathbb{R}^{n_q}$, where $V_q := \text{span}(B_q, A_q V_p)$. Then $V_q$ and $V_q$ are spanned by $V_q = \text{Orth}(V_q)$ and $W_q = \text{Orth}(V_q)$, respectively. We note that $V_q$ and $W_q$ must be of the same dimension. Thus substituting $\xi_q = V_p \xi_p$ and $\xi_q = V_q \xi_q$ into (27) and left multiplying (27a) by $W_q^T$, we obtain the reduced-order model of the algebraic part (27) given by

$$E_q \xi_q = A_q V_p \xi_p + B_q u,$$  \hfill (30a)

$$y_q = C_q^T \xi_q.$$  \hfill (30b)

where $E_q = W_q^T E_q V_q \in \mathbb{R}^{r_2 \times r_2}$, $A_q = W_q^T A_q V_p \in \mathbb{R}^{r_1 \times r_1}$, $B_q = W_q^T B_q \in \mathbb{R}^{r_2 \times m}$ and $C_q = V_q^T C_q \in \mathbb{R}^{r_2 \times l}$, where $r_1$ is the reduced dimension of the differential part and $r_2 = \dim(V_q) = \dim(V_q)$ which is equal to the reduced dimension of the algebraic part. Hence recombining (28) and (30) we obtain the implicit-IMOR reduced-order model for (3) given by

$$E_r \xi_r = \tilde{A}_r \xi_r + \tilde{B}_r u,$$  \hfill (31a)

$$y_r = C_r^T \xi_r.$$  \hfill (31b)

where $E_r = W^T E \tilde{V}$, $\tilde{A}_r = W^T \tilde{A} \tilde{V}$, $\tilde{B}_r = W^T B \tilde{C}_r = \tilde{V}^T C_r$, where $\tilde{W} = \begin{bmatrix} V_p & 0 \\ 0 & V_q \end{bmatrix}$, $\tilde{V} = \begin{bmatrix} V_p & 0 \\ 0 & V_q \end{bmatrix}$. We note that it can easily be proved that the implicit-IMOR methods also preserves the goals of the MOR depending on the MOR method used to reduce the differential part and moreover it preserve the index of the DAE. For instance if PRIMA method is used to reduce the differential part then the passivity can be guaranteed if the original matrices satisfies properties suggested in [7]. We note the projectors and their corresponding bases are numerically feasible and can be computed using the LU based routine proposed in [9] for the case of sparse matrices. But for the case of dense matrices SVD based routines have to be used.
the fact that $\tilde{A}$ does not involve matrix inversion for the case of implicit splitting method. Explicit splitting method has a sparser matrix $\tilde{E}$ just because $E_p$ and $E_q$ are always identity matrices. Another advantage of implicit splitting over explicit splitting method that it partially preserves the original structure of matrix pencil $([E, A])$ which is very important in the electric networks community.

For convenience we use the last example in Tab I to compare the reduced-order models obtained using the IIMOR and IMOR methods which is a SISO model but the IMOR method can as well be used on the MIMO models. Using $s_0 = 10$ as the expansion point, we were able to reduce the differential and algebraic part of decoupled system to 375 and 100, respectively using the IMOR and IMOR method. Thus the dimension of the reduced-order models is 475. In Fig. 2, we compare the magnitude of the transfer function of the IMOR and IMOR reduced -order models. We observe that both reduced-order model coincides with that of the original models. In Fig. 3, we compare the their respective approximation error. We can see that IMOR reduced-order model is more accurate than the IMOR method. However, even if the IMOR method may be more accurate than the IIMOR method it is computationally expensive. Therefore, we need to trade off between accuracy and computational costs while using the two methods.

<table>
<thead>
<tr>
<th>Power system</th>
<th>Implicit splitting method</th>
<th>Explicit splitting method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$nnz(E)$</td>
<td>$nnz(A)$</td>
</tr>
<tr>
<td>Juba5723</td>
<td>1810871</td>
<td>352416</td>
</tr>
<tr>
<td>Basurn5727</td>
<td>1810871</td>
<td>352416</td>
</tr>
<tr>
<td>zinga3012</td>
<td>3012</td>
<td>5833194</td>
</tr>
<tr>
<td>BIPS1997</td>
<td>201542</td>
<td>448799</td>
</tr>
</tbody>
</table>

Fig. 2. Comparison of Transfer function

VI. CONCLUSION

In conclusion, we have have proposed the IIMOR method which an implicit version of the IMOR method. This method is computationally cheaper than the IMOR method. Also it partially preserves the original structure of the DAE system. The IMOR method may be more accurate than the IIMOR method but it is computationally expensive to be use to reduce large RC/RLC networks. Hence IIMOR method is the best optional. Finally, the IIMOR method can be extended to systems with higher tractability index. This will be the topic of a forthcoming paper.

ACKNOWLEDGMENT

This work was funded by The Netherlands Organisation for Scientific Research (NWO).

REFERENCES